Derivation of an eigenvalue probability density function relating to the Poincaré disk

Peter J Forrester[1] **and Manjunath Krishnapur**[2]

[1] Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia
[2] Department of Mathematics, University of Toronto, ON M5S 2E4, Canada

E-mail: p.forrester@ms.unimelb.edu.au

Abstract

A result of Zyczkowski and Sommers (2000 *J. Phys. A: Math. Gen.* **33** 2045–57) gives the eigenvalue probability density function for the top $N \times N$ sub-block of a Haar distributed matrix from $U(N + n)$. In the case $n \geqslant N$, we rederive this result, starting from knowledge of the distribution of the sub-blocks, introducing the Schur decomposition and integrating over all variables except the eigenvalues. The integration is done by identifying a recursive structure which reduces the dimension. This approach is inspired by an analogous approach which has been recently applied to determine the eigenvalue probability density function for random matrices $A^{-1}B$, where A and B are random matrices with entries standard complex normals. We relate the eigenvalue distribution of the sub-blocks to a many-body quantum state, and to the one-component plasma, on the pseudosphere.

1. Introduction

The plane, sphere and pseudosphere are special geometries in a number of related many-body statistical systems. In particular, the lowest Landau level wavefunction for quantum particles in a magnetic field, the equilibrium statistical mechanics of a one-component plasma at a special value of the coupling and the eigenvalue probability density function for certain random matrix ensembles all give rise to a special solvable state in these geometries. In regard to the first of these, regarding the geometries as surfaces in appropriate 3-spaces, and imposing a perpendicular magnetic field, the solvable state is the ground state wavefunction, in that it can be written in an explicit factorized (Jastrow) form and the correlations can be expressed explicitly as determinants [4, 5]. The absolute value squared of these wavefunctions

has the interpretation as the Boltzmann factor for a one-component plasma, confined to the corresponding surface, and interacting at the special value of the coupling $\Gamma := q^2/k_B T = 2$ [2, 15]. And the Boltzmann factors, after projecting onto the plane via an appropriate stereographic projection in the case of the sphere and pseudosphere, allow for realizations as the eigenvalue probability density function for three particular random matrix ensembles [12, 16, 23]. The determinant form for the correlations in the quantum problem carries over to the plasma and eigenvalue distributions, so these many-body systems are also solvable. Although not to be addressed here, we also mention that these same geometries play a special role in relation to the study of the zeros of random polynomials [9, 18].

It is the purpose of the present paper to contribute to the theory of the random matrix ensemble relating to the pseudosphere, or more precisely to the Poincaré disk. Specifically, our interest is in the eigenvalue distribution of the ensemble of the top $N \times N$ sub-block of Haar distributed matrices in $U(N + n)$ (due to the invariance of Haar distributed unitary matrices with respect to left and right multiplication by permutation matrices, in fact sub-blocks formed by the entries of any N rows and N columns will have the same distributions as the top $N \times N$ sub-block). This ensemble has shown itself in a number of recent works in random matrix theory [8, 11, 17, 21]. We present a new derivation of a result due to Zyczkowski and Sommers [23], giving that the eigenvalue probability density function is proportional to

$$\prod_{l=1}^{N}(1 - |z_l|^2)^{n-1}\chi_{|z_l|<1} \prod_{1\leqslant j<k\leqslant N} |z_k - z_j|^2, \tag{1.1}$$

(here $\chi_A = 1$ if A is true, and $\chi_A = 0$ otherwise). Our proof, although applying only to the case when $n \geqslant N$, is arguably simpler than the original. Moreover, our derivation unifies the case of the sphere and pseudosphere, by showing that a recently formulated approach to the eigenvalue probability density function of the random matrix ensemble corresponding to the sphere [13], can be adapted to the case of the pseudosphere.

2. The pseudosphere and plasma system

The pseudosphere refers to the two-dimensional hyperbolic space with the constant negative Gaussian curvature $\kappa = -1/a^2$ [1]. It is naturally embedded in the three-dimensional Minkowski space with coordinates (y_0, y_1, y_2) and line element

$$(ds)^2 = -(dy_0)^2 + (dy_1)^2 + (dy_2)^2. \tag{2.1}$$

The pseudosphere is defined as the upper branch of the equation

$$-y_0^2 + y_1^2 + y_2^2 = -a^2$$

and can be parametrized by

$$y_0 = a \cosh \tau, \qquad y_1 = a \sinh \tau \cos \phi, \qquad y_2 = a \sinh \tau \sin \phi.$$

In terms of this parametrization the line element (2.1) reads

$$(ds)^2 = a^2 (d\tau)^2 + a^2 \sinh^2 \tau \, (d\phi)^2. \tag{2.2}$$

Furthermore, the geodesic distance between (τ, ϕ) and (τ', ϕ') is such that

$$\cosh \frac{s}{a} = \cosh \tau \cosh \tau' - \sinh \tau \sinh \tau' \cos(\phi - \phi'), \tag{2.3}$$

while the volume element corresponding to the surface area is

$$dS = a^2 \sinh \tau \, d\tau \, d\phi. \tag{2.4}$$

With $z := x + iy$, the pseudosphere is projected onto the Poincaré disk via the stereographic projection

$$z = 2a \tanh \frac{\tau}{2} e^{i\phi}, \qquad |z| < 2a \tag{2.5}$$

(equivalently $(y_0, y_1, y_2) \mapsto 2a(y_1/y_0, y_2/y_0)$) and then

$$dS = \frac{dx \, dy}{(1 - |z|^2/4a^2)^2}. \tag{2.6}$$

Note that with z expressed in polar form, (2.5) is equivalent to writing

$$r = 2a \tanh \frac{\tau}{2}, \qquad |z| < 2a.$$

In terms of the coordinates (τ, ϕ) the Laplacian Δ is computed from (2.2) as

$$\Delta = \frac{1}{a^2} \left(\frac{1}{\sinh \tau} \frac{\partial}{\partial \tau} \sinh \tau \frac{\partial}{\partial \tau} + \frac{1}{\sinh^2 \tau} \frac{\partial^2}{\partial \phi^2} \right).$$

Alternatively, in terms of the coordinates (r, ϕ) the Laplacian reads

$$\Delta = \left(1 - \frac{r^2}{4a^2} \right)^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right). \tag{2.7}$$

From the Laplacian we can specify the Coulomb potential on the pseudosphere. In fact, two potentials Φ and $\tilde{\Phi}$ will be defined. With s_{jk} denoting the geodesic separation between particles j and k, and Δ_j the Laplacian with respect to particle j, the first is specified as the solution of the Poisson equation

$$\Delta_j \Phi(s_{jk}) = -2\pi \delta(s_{jk}), \tag{2.8}$$

subject to the boundary condition that $\Phi(s) \to 0$ as $s \to \infty$, and the second as the solution of the Poisson equation

$$\Delta_j \tilde{\Phi}(s_{jk}) = -2\pi \delta(s_{jk}) - \frac{1}{2a^2}. \tag{2.9}$$

In both (2.8) and (2.9) $\delta(s_{jk})$ denotes the delta function relative to the volume form (2.4). The Poisson equation (2.8) corresponds to the potential at (τ_j, ϕ_j) due to a unit charge at (τ_k, ϕ_k), while (2.9) corresponds to the potential at (τ_j, ϕ_j) due to a unit charge at (τ_k, ϕ_k) and a uniform smeared out charge of the same sign and density $1/4\pi a^2$. The solution of (2.8) is readily verified to be

$$\Phi(s_{jk}) = -\log \tanh \frac{s_{jk}}{2} = -\log \left| \frac{(z_j - z_k)/2a}{1 - z_j \bar{z}_k/4a^2} \right|, \tag{2.10}$$

and that of (2.9) to be

$$\tilde{\Phi}(s_{jk}) = -\log \sinh \frac{s_{jk}}{2} = -\log \left(\frac{|z_j - z_k|/2a}{(1 - (r_j/2a)^2)^{1/2} (1 - (r_k/2a)^2)^{1/2}} \right). \tag{2.11}$$

Choosing either (2.10) or (2.11), we want to form a one-component plasma system, consisting of N mobile unit charges, and a smeared out uniform background charge density of net charge $-\eta$. In the large N limit, we know from [15] that the bulk equilibrium statistical mechanics is the same for both. However, our interest is in finite N, and to obtain an analogy with a random matrix eigenvalue probability density function it is necessary to base the plasma system on the potential (2.11). Since (2.11) corresponds to (2.9), to get a net background charge density $-\eta$ we must also cancel the smeared out positive charge implied by the constant

term therein, and thus impose a smeared out charge density $-\eta - N/4\pi a^2$. This will interact with the particles to create a potential $V(r)$, which must satisfy the Poisson equation

$$\Delta V(r) = 2\pi(\eta + N/4\pi a^2).$$

Choosing for convenience $V(0) = 0$ (the constant terms in the total potential are of no interest to us), and using the form (2.8) of Δ, the solution of this equation is seen to be

$$V(r) = -(2\pi\eta a^2 + N/2)\log\left(1 - \frac{r^2}{4a^2}\right). \tag{2.12}$$

The total potential energy U, excluding constants, is obtained by summing the potential (2.11) over all pairs, and adding to this $\sum_{j=1}^{N} V(r_j)$. We thus have [15]

$$U = -\sum_{1\leqslant j<k\leqslant N}\log(|z_k - z_j|/2a) - (2\pi\eta a^2 + 1/2)\sum_{j=1}^{N}\log\left(1 - r_j^2\big/4a^2\right)$$

implying that the Boltzmann factor is such that

$$\mathrm{e}^{-\beta U}\,\mathrm{d}S_1\cdots\mathrm{d}S_N \propto \prod_{l=1}^{N}(1 - |z_l|^2/4a^2)^{(2\pi\eta a^2 + 1/2)\beta - 2}\prod_{1\leqslant j<k\leqslant N}|z_k - z_j|^\beta\,\mathrm{d}x_1\,\mathrm{d}y_1\cdots\mathrm{d}x_N\,\mathrm{d}y_N. \tag{2.13}$$

3. Magnetic analogy

It is remarked in [15] that with $l := \sqrt{\hbar c/eB}$ the magnetic length

$$\psi_j(z) = \left(1 - \frac{|z|^2}{4a^2}\right)^{(a/l)^2}\bar{z}^j, \qquad j = 0, 1, \ldots \tag{3.1}$$

are, up to normalization, the lowest Landau level eigenstates for a quantum particle on the pseudosphere, in the presence of a constant perpendicular magnetic field B, and projected onto the Poincaré disk. Forming a Slater determinant out of the first N of these gives an N-body state ψ such that $|\psi|^2$ is proportional to (2.13) with

$$\beta = 2, \qquad 4\pi\eta a^2 + 1 = 2(a/l)^2. \tag{3.2}$$

We take this opportunity to give some details on how (3.1) comes about (see also [7]).

Let $\vec{r}$ denote the unit vector perpendicular to the pseudosphere in Minkowski space. By the formal analogy with the surface of a sphere with radius R and polar coordinates (θ, ϕ) as specified by

$$a \leftrightarrow iR, \qquad \tau \leftrightarrow i\theta,$$

one sees that the vector potential

$$\vec{A} = \left(0, 0, \frac{Ba}{\sinh\tau}(1 - \cosh\tau)\right) \tag{3.3}$$

is such that $\nabla \times \vec{A} = B\hat{\vec{r}}$ and so corresponds to the sought perpendicular magnetic field. Further, it follows similarly that the Hamiltonian

$$H := \frac{1}{2m}\left(-i\hbar\nabla + \frac{e}{c}\vec{A}\right)^2$$

is such that

$$H = \frac{\hbar^2}{2ma^2}\left(-\frac{1}{\sinh\tau}\frac{\partial}{\partial\tau}\left(\sinh\tau\frac{\partial}{\partial\tau}\right) + \frac{1}{\sinh^2\tau}\left(\frac{\partial}{\partial\phi} - \left(\frac{a}{l}\right)^2(\cosh\theta - 1)^2\right)\right) \tag{3.4}$$

4

(of course in writing down (3.3) and (3.4) knowledge of the corresponding results for the sphere is assumed [22]). The change of variables (2.5) so as to project to the Poincaré disk gives that (3.4) reads

$$H = \frac{\hbar^2}{2m}\left(-4l^2\left(1 - \frac{|z|^2}{4a^2}\right)\frac{\partial^2}{\partial z \partial \bar{z}} + \left(1 - \frac{|z|^2}{4a^2}\right)\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) + \frac{|z|^2}{4l^2}\right). \tag{3.5}$$

Following [5], with $\omega_c := eB/mc$, we note that the projected Hamiltonian (3.5) is of the form

$$H = -\frac{\hbar\omega_c}{2}\left(\left(\frac{\partial^2\Phi}{\partial z \partial \bar{z}}\right)^{-1}\left(\frac{\partial}{\partial \bar{z}} - \frac{\partial\Phi}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial\Phi}{\partial z}\right) - 1\right) \tag{3.6}$$

with

$$\Phi = -\left(\frac{a}{l}\right)^2 \log\left(1 - \frac{|z|^2}{(2a)^2}\right).$$

But (3.6) is such that

$$e^\Phi H e^{-\Phi} = -\frac{\hbar\omega_c}{2}\left(\left(\frac{\partial^2\Phi}{\partial z \partial \bar{z}}\right)^{-1}\left(\frac{\partial}{\partial z} - 2\frac{\partial\Phi}{\partial \bar{z}}\right)\frac{\partial}{\partial z} - 1\right)$$

and so we are led to the result that any function of the form $e^{-\Phi}f(\bar{z})$ with f analytic in $\bar{z}$ belongs to the lowest Landau level (i.e. is an eigenfunction of H with the eigenvalue $\hbar\omega_c/2$). To further specify f, one notes that

$$J := z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$$

commutes with H as defined by (3.6) for general $\Phi = \Phi(|z|^2)$, and seeks simultaneous eigenfunctions of H and J. Such eigenfunctions are given by

$$e^{-\Phi}\bar{z}^j, \qquad j = 0, 1, \ldots$$

and (3.1) follows.

Now if all the particles are fermions, the ground state ψ is formed out of a Slater determinant of the single particles states,

$$\psi(z_1, \ldots, z_N) = \frac{1}{C}\det[\psi_{j-1}(z_j)]_{j,k=1,\ldots,N}$$

$$= \frac{1}{C}\prod_{l=1}^N\left(1 - \frac{|z_l|^2}{4a^2}\right)^{(a/l)^2}\prod_{1\leqslant j<k\leqslant N}(\bar{z}_k - \bar{z}_j), \tag{3.7}$$

where C is an appropriate normalization, and the second equality follows upon use of the Vandermonde determinant identity. The modulus squared is indeed identified with the case (3.2) of (2.13).

4. Random matrix ensembles

4.1. The spherical ensemble

The eigenvalue probability density function for matrices $Y = A^{-1}B$, with A and B random $N \times N$ matrices consisting of independent complex Gaussian entries, turns out to be a one-component plasma on the sphere. This spherical analog of (1.1) was proved in [17]. Below, we give a simplified and unified approach that works for the sphere as well as the pseudosphere.

We first sketch the proof for the sphere (more details may be found in the forthcoming book [13]). The first step is to determine the distribution of Y.

Proposition 1. *The distribution of Y is proportional to*

$$\frac{1}{\det(\mathbb{I}_N + YY^\dagger)^{2N}}. \tag{4.1}$$

Proof. With X a complex matrix and $(\mathrm{d}X)$ denoting the wedge product of the differentials of the real and imaginary parts of its elements, by noting

$$(\mathrm{d}B) = |\det A|^{2N}(\mathrm{d}Y) \tag{4.2}$$

it follows that the joint density of A and Y is proportional to

$$|\det A|^{2N}\,\mathrm{e}^{-\mathrm{Tr}\{A(\mathbb{I}_N+YY^\dagger)A^\dagger\}}.$$

The distribution of Y is obtained by integrating this over A. For this we change variables $C = A(\mathbb{I}_N + YY^\dagger)^{1/2}$ and by an appropriate analog of (4.2) and integration over C, we obtain the density of Y to be

$$\int \frac{|\det(C)|^{2N}\,\mathrm{e}^{-\mathrm{Tr}(CC^\dagger)}}{\det(\mathbb{I}_N + YY^\dagger)^N}\,\frac{\mathrm{d}C}{\det(\mathbb{I}_N + YY^\dagger)^N},$$

from which (4.1) results.   □

The second and final step is to use knowledge of the distribution (4.1) to proceed to compute the eigenvalue probability density function.

Proposition 2. *[16] The eigenvalue probability density function for Y is proportional to*

$$\prod_{l=1}^{N} \frac{1}{(1 + |z_l|^2)^{N+1}} \prod_{1 \leqslant j < k \leqslant N} |z_k - z_j|^2. \tag{4.3}$$

Proof. According to the Schur decomposition, Y can be written as

$$Y = U T_N U^{-1}, \tag{4.4}$$

where U is a unitary matrix, and T_N is an upper triangular matrix in which the diagonal entries, denoted by $\{z_j\}_{j=1,\dots,N}$, are the eigenvalues of Y. Furthermore, we know that (see appendix 35 of [19] or section 6.3 of [13])

$$(\mathrm{d}Y) = \prod_{1 \leqslant j < k \leqslant N} |z_k - z_j|^2 (U^\dagger \mathrm{d}U) \wedge_{j=1}^{N} \mathrm{d}z_j (\mathrm{d}\tilde{T}_N), \tag{4.5}$$

where $\tilde{T}_N$ denotes the strictly upper portion of T_N. Noting from (4.4) that $YY^\dagger = U T_N T_N^\dagger U^\dagger$ shows that (4.1) is independent of U. The factorization of the U dependence in (4.5) then implies that the eigenvalue distribution is obtained from (4.1) by integrating out the dependence of $\tilde{T}_N$ and is thus proportional to

$$\prod_{1 \leqslant j < k \leqslant N} |z_k - z_j|^2\, I_N(z_1, \dots, z_N), \qquad I_N(z_1, \dots, z_N) := \int \frac{1}{\det\left(\mathbb{I}_N + T_N T_N^\dagger\right)^{2N}}\,(\mathrm{d}\tilde{T}_N). \tag{4.6}$$

Following [13], we will compute the integral I_N by deducing a recurrence for the integrals

$$I_{n,p}(z_1, \dots, z_n) := \int \frac{1}{\det\left(\mathbb{I}_n + T_n T_n^\dagger\right)^p}\,(\mathrm{d}\tilde{T}_n), \tag{4.7}$$

where $p \geqslant n$ (this ensures convergence). Analogous strategies can be found in [6, 10, 14]. For this purpose, let $\vec{u}$ denote the last column of $\tilde{T}_n$. We can then write

$$\mathbb{I}_n + T_n T_n^\dagger = \begin{bmatrix} \mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger + \vec{u}\vec{u}^\dagger & z_n \vec{u} \\ \bar{z}_n \vec{u}^\dagger & 1 + |z_n|^2 \end{bmatrix}, \tag{4.8}$$

showing

$$\det \left(\mathbb{I}_n + T_n T_n^\dagger \right) = (1 + |z_n|^2) \det \left(\mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger + \frac{\vec{u}\vec{u}^\dagger}{1 + |z_n|^2} \right).$$

Noting that the final term in the determinant is a matrix of rank 1, this latter formula can be further reduced to read

$$\det \left(\mathbb{I}_n + T_n T_n^\dagger \right) = (1 + |z_n|^2) \det \left(\mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger \right) \left(1 + \frac{1}{1 + |z_n|^2} \vec{u}^\dagger \left(\mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger \right)^{-1} \vec{u} \right). \tag{4.9}$$

Hence, by first writing

$$I_{n,p}(z_1, \ldots, z_n) = \int (d\tilde{T}_{n-1}) \int (d\vec{u}) \frac{1}{\det \left(\mathbb{I}_n + T_n T_n^\dagger \right)^p}$$

we can substitute (4.9) to obtain

$$I_{n,p}(z_1, \ldots, z_n) = \frac{1}{(1 + |z_n|^2)^p} \int (d\tilde{T}_{n-1}) \frac{1}{\det \left(\mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger \right)^p}$$
$$\times \int (d\vec{u}) \left(1 + \frac{1}{1 + |z_n|^2} \vec{u}^\dagger \left(\mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger \right)^{-1} \vec{u} \right)^{-p}. \tag{4.10}$$

Making the change of variables

$$\vec{v} = (1 + |z_n|^2)^{-1/2} \left(\mathbb{I}_{n-1} + T_{n-1} T_{n-1}^\dagger \right)^{-1/2} \vec{u}$$

we see from this that

$$I_{n,p}(z_1, \ldots, z_n) = \frac{C_{n,p}}{(1 + |z_n|^2)^{p-n+1}} I_{n-1,p-1}(z_1, \ldots, z_{n-1}) \tag{4.11}$$

where

$$C_{n,p} = \int \frac{(d\vec{v})}{(1 + \vec{v}^\dagger \vec{v})^p}. \tag{4.12}$$

Iterating (4.11) with $n = 2N$, $p = N$ shows

$$I_N(z_1, \ldots, z_N) \propto \prod_{l=1}^{N} \frac{1}{(1 + |z_l|^2)^{N+1}}$$

and this substituted in (4.6) gives (4.3). $\qquad \square$

We remark that the name spherical ensemble comes about because upon making the stereographic projection from the plane to the sphere,

$$z = 2R \tan \frac{\theta}{2} e^{i\phi}$$

(cf (2.5)) transforms (4.3) to the form

$$\prod_{1 \leqslant j < k \leqslant N} |\vec{r}_j - \vec{r}_k|^2$$

where $\vec{r}_j$ is the vector in $\mathbb{R}^3$ corresponding to the point (R, θ_j, ϕ_j) on the sphere.

4.2. Truncated unitary matrices—the pseudosphere ensemble

Working now with truncated unitary matrices, we would like to repeat the steps which lead to establishing the eigenvalue probability density function (4.3). The first task then is to specify the distribution of the top $N \times N$ sub-block of Haar distributed matrices from $U(N + n)$. Our working requires that $n \geqslant N$.

Proposition 3. *[3] Let the top $N \times N$ sub-matrices in question be denoted by Q_N. The probability density of Q_N is proportional to*

$$\det \left(\mathbb{I}_N - Q_N Q_N^\dagger \right)^{n-N}, \tag{4.13}$$

supported on $Q_N Q_N^\dagger < 1$ (i.e. matrices Q_N with all singular values less than 1).

Proof. Following [8], we give a variation of the proof given in [3]. The proof relies on a standard result in random matrix theory giving a construction of a particular Jacobi ensemble in terms of Wishart matrices [7, 20]. With c and d being independent $N \times N$ and $n \times N$ complex Gaussian random matrices, respectively (all elements i.i.d. standard complex normals), let $C = c^\dagger c$, $D = d^\dagger d$. We require the fact that for $n \geqslant N$ the probability density of

$$J := (C + D)^{-1/2} C (C + D)^{-1/2} \tag{4.14}$$

is proportional to

$$\det(\mathbb{I}_N - J)^{n-N}, \tag{4.15}$$

supported on $J < 1$.

   Let U denote an $(N + n) \times (N + n)$ Haar distributed unitary matrix. Let W denote the $N \times (N+n)$ sub-matrix formed from the first N rows, and $\tilde{W}$ denote the $n \times (N+n)$ sub-matrix formed from the final n rows, so U has the block decomposition

$$U = \begin{bmatrix} W \\ \tilde{W} \end{bmatrix}.$$

Further, let X be an $(N + n) \times N$ Gaussian matrix of independent standard complex normals. From the facts that WX is then also distributed as an $N \times N$ Gaussian matrix of standard complex normals, and $\tilde{W}X$ as an $n \times N$ Gaussian matrix of standard complex normals, and that $WX, \tilde{W}X$ are independent, we can take $c = WX$ and $d = \tilde{W}X$ in (4.14) to conclude from the above result that

$$J = (X^\dagger X)^{-1/2} X^\dagger W^\dagger W X (X^\dagger X)^{-1/2} \tag{4.16}$$

has probability density (4.15).

   Introduce the singular value decomposition by writing

$$X = U_1 \Lambda U_2$$

for U_1 an $(N+n) \times (N+n)$ unitary matrix, U_2 an $N \times N$ unitary matrix, and Λ an $(N+n) \times N$ matrix with diagonal entries (that is entries (i, i), for $i \leqslant N$) equal to the positive square root of the eigenvalues of $X^\dagger X$. We then have

$$X(X^\dagger X)^{-1/2} = [U_1]_{N+n,N} U_2$$

where the notation $[Z]_{p,q}$ denotes the top $p \times q$ sub-block of Z. Using this in (4.16) shows

$$J = U_2^\dagger [U_1^\dagger]_{N,N+n} W^\dagger W [U_1]_{N+n,N} U_2.$$

But the distribution of J is unchanged under conjugation by unitary matrices, and in particular U_2. Applying this conjugation, we see that

$$J \text{ has the same distribution as } [U_1^\dagger]_{N,N+n} W^\dagger W [U_1]_{N+n,N}. \tag{4.17}$$

Since U_1 and U are independent, and U has the Haar distribution, UU_1 has the same distribution as U and hence, W has the same distribution as $[WU_{1,1} \; WU_{1,2}]$, where we have written $U_{1,1} = [U]_{N+n,N}$ and $U_{1,2}$ denotes the last n columns of U. From this, it immediately follows that $[W^\dagger W]_{N,N}$ has the same distribution as $\left[U_1^\dagger\right]_{N,N+n} W^\dagger W[U_1]_{N+n,N}$. Together with (4.17), we conclude that $[W^\dagger W]_{N,N}$ has the same distribution as J.

This tells us that the probability density of $Q_N^\dagger Q_N$ is proportional to (4.13). The probability density of Q_N itself, being the top $N \times N$ block of U, $U \in U(N+n)$, must be invariant under conjugation by $S \in U(N)$ and under complex conjugation, and so be a function of (the trace of) $Q_N^\dagger Q_N$. But in general for a $p \times q$ ($p \geqslant q$) complex matrix Y with the probability density f of the form $f = f\left(Q_N^\dagger Q_N\right)$, the probability density for $Y^\dagger Y$ is proportional to $(\det Y^\dagger Y)^{p-q} f(Y^\dagger Y)$. Here $p = q = N$, so we conclude that (4.13) also holds for the probability density of Q_N.

$\square$

Knowing that Q_N is distributed according to (4.13), the task now is to introduce the Schur decomposition of Q_N, and integrate out all variables but the eigenvalues. This can be done using the same recursive reduction of the dimension of the integral as used in the proof of proposition 2, and the sought probability density function thus obtained.

Proposition 4. *Let $n \geqslant N$. Then, the eigenvalue probability density function of Q_N is proportional to (1.1).*

Proof. Because Q_N is a sub-block of a unitary matrix, all eigenvalues have modulus less than 1. Introducing the Schur decomposition (4.4) of Q_N in (4.13), and recalling the change of variables formula (4.5), we see that the eigenvalue probability density function is proportional to

$$\prod_{1 \leqslant j < k \leqslant N} |z_k - z_j|^2 \, J_N(z_1, \ldots, z_N), \qquad J_N(z_1, \ldots, z_N) := \int \det\left(\mathbb{I}_N - T_N T_N^\dagger\right)^{n-N} (\mathrm{d}\tilde{T}_N),$$

(4.18)

where again $\tilde{T}_N$ denotes the strictly upper triangular portion of T_N.

To compute J_N by recurrence we introduce, for $p \geqslant 0$,

$$J_{m,p}(z_1, \ldots, z_m) := \int \det\left(\mathbb{I}_m - T_m T_m^\dagger\right)^p (\mathrm{d}\tilde{T}_m)$$

(4.19)

(cf (4.7)). Analogous to (4.8), with $\vec{u}$ denoting the last column of $\tilde{T}_N$, we can write

$$\mathbb{I}_m - T_m T_m^\dagger = \begin{bmatrix} \mathbb{I}_{m-1} - T_{m-1} T_{m-1}^\dagger - \vec{u}\vec{u}^\dagger & -z_m \vec{u} \\ -\bar{z}_m \vec{u}^\dagger & 1 - |z_m|^2 \end{bmatrix}.$$

Working now as in deriving (4.9) from (4.8) shows

$$\det\left(\mathbb{I}_m - T_m T_m^\dagger\right) = (1 - |z_m|^2) \det\left(\mathbb{I}_{m-1} - T_{m-1} T_{m-1}^\dagger\right)$$
$$\times \left(1 - \frac{1}{1 - |z_m|^2} \vec{u}^\dagger \left(\mathbb{I}_{m-1} - T_{m-1} T_{m-1}^\dagger\right)^{-1} \vec{u}\right).$$

(4.20)

Substituting (4.20) in (4.19) and proceeding now as in the derivation of (4.11) from (4.10) shows

$$J_{m,p}(z_1, \ldots, z_m) = C_{m,p}(1 - |z_m|^2)^{m+p-1} J_{m-1,p+1}(z_1, \ldots, z_{m-1})$$

where $C_{m,p}$ is again given by (4.12). Iterating this equation m times, we get

$$J_{m,p}(z_1, \ldots, z_m) = C_{m,p} \prod_{k=1}^{m} (1 - |z_k|^2)^{m+p-1}.$$

(4.21)

9

Applying (4.21) with $m = N$, $p = n - N$ shows

$$J_N(z_1, \ldots, z_N) \propto \prod_{l=1}^{N} (1 - |z_l|^2)^{n-1}$$

and this substituted in (4.18) gives (1.1). ∎

Acknowledgments

References

[1] Balazs N L and Voros A 1986 Chaos on the pseudosphere *Phys. Rep.* **143** 109–240
[2] Caillol J M 1981 Exact results for a two-dimensional one-component plasma on a sphere *J. Phys. Lett.* (Paris) **42** L245–7
[3] Collins B 2005 Product of random projections, Jacobi ensembles and universality problems arising from free probability *Probab. Theory Relat. Fields* **133** 315–44
[4] Comtet A 1987 On the Landau levels on the hyperbolic space *Ann. Phys.* **173** 185–209
[5] Dunne G V 1992 Hilbert space for charged particles in perpendicular magnetic fields *Ann. Phys.* **215** 233–63
[6] Edelman A, Kostlan E and Shub M 1994 How many eigenvalues of random matrix are real? *J. Am. Math. Soc.* **7** 247–67
[7] Forrester P J 2009 *Log-Gases and Random Matrices* (Princeton, NJ: Princeton University Press) at press
[8] Forrester P J 2006 Quantum conductance problems and the Jacobi ensemble *J. Phys. A: Math. Gen.* **39** 6861–70
[9] Forrester P J and Honner G 1999 Exact statistical properties of the zeros of complex random polynomials *J. Phys. A: Math. Gen.* **32** 2961–81
[10] Fyodorov Y V and Khoruzhenko B A 2007 On absolute moments of characteristic polynomials of a certain class of complex random matrices *Commun. Math. Phys.* **273** 561–99
[11] Fyodorov Y V and Khoruzhenko B A 2007 A few remarks on colour-flavour transformations, truncation of random unitary matrics, Berezin reproducing kernels and Selberg type integrals *J. Phys. A: Math. Theor.* **40** 669–700
[12] Ginibre J 1965 Statistical ensembles of complex, quaternion, and real matrices *J. Math. Phys.* **6** 440–9
[13] Hough J B, Krishnapur M, Peres Y and Virág B 2008 *Zeros of Gaussian Analytic Functions and Determinantal Point Processes* (Providence, RI: American Mathematical Society)
[14] Hua L K 1963 *Analysis of Functions of Several Complex Variables in the Classical Domains* (Providence, RI: American Mathematical Society)
[15] Jancovici B and Téllez G 1998 Two-dimensional Coulomb systems on a surface of constant negative curvature *J. Stat. Phys.* **91** 953–77
[16] Krishnapur M 2006 Zeros of random analytic functions *PhD Thesis* University of California, Berkeley (arXiv:math.Pr/0607504)
[17] Krishnapur M 2009 From random matrices to random analytic functions *Ann. Probab.* **37** 314–46
[18] Leboeuf P 2000 Random matrices, random polynomials and coulomb systems *J. Physique IV* **10** 5–45
[19] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[20] Muirhead R J 1982 *Aspects of Multivariable Statistical Theory* (New York: Wiley)
[21] Wei Y and Fyodorov Y V 2008 On the mean density of complex eigenvalues for an ensemble of random matrices with prescribed singular values *J. Phys. A: Math. Theor.* **41** 502001
[22] Wu T T and Yang C N 1976 Dirac monopole without string: monopole harmonics *Nucl. Phys.* B **107** 365–80
[23] Zyczkowski K and Sommers H-J 2000 Truncations of random unitary matrices *J. Phys. A: Math. Gen.* **33** 2045–57